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INFORMATION FUSION IN POLAR COORDINATES

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ABSTRACT

Many sensing modalities used in robotics collect information in polar coordinates. For mobile robots and autonomous vehicles these modalities include radar, sonar, and laser range finders. In the context of medical robotics, ultrasound imaging and CT both collect information in polar coordinates. Moreover, every sensing modality has associated noise. Therefore when the position of a point in space is estimated in the reference frame of the sensor, that position is replaced by a probability density expressed in polar coordinates. If the sensor moves from one location to another and the same point is sensed, then the two associated probabilities can be "fused" together to obtain a better estimate than either one individually. Here we derive the equations for this fusion process in polar coordinates. The result involves the computation of integrals of three Bessel function. We derive new recurrence relations for the efficient computation of these Bessel-functon integrals to aid in the information-fusion process.

INTRODUCTION

The definition of information fusion is proposed as: "Information fusion is the study of efficient methods for automatically or semi-automatically transforming information from different sources and different points in time into a representation that provides effective support for human or automated decision making [1]. The information provided by sensors is always impreciseness that is affected by associated noise in the measurements. Information fusion algorithms should be able to exploit the redundancy and imprecise information to reduce effects of noise.

Let 0, 1, 2 denote three points in the plane and let $\mathbf{x}_{ij}$ denote the relative position vector from point i to point j. The length of this vector is $r_{ij} = \|\mathbf{x}_{ij}\| \doteq \sqrt{\mathbf{x}_{ij} \cdot \mathbf{x}_{ij}}$. Relative to the x-axis of a world coordinate system, the vector $\mathbf{x}_{ij}$ makes a counterclockwise measured angle ϕ_{ij} as shown in Figure 1. Suppose that point 2 is observed by a sensor at point 0 and at point 1. The sensors at locations 0 and 1 may be different sensors that are fixed at these locations, or they may be the same sensor that has moved between these locations. Associated with these sensors are measurement probability densities $f_{02}(\mathbf{x}_{02})$ and $f_{12}(\mathbf{x}_{12})$ of the point at position 2.

Recall that in general probability densities $f : X \to \mathbb{R}$ satisfy the conditions

$$f(\mathbf{x}) \geq 0 \text{ and } \int_X f(\mathbf{x})\, d\mathbf{x} = 1$$

and in our case $X = \mathbb{R}^2$ and $d\mathbf{x}$ is the Lebesgue measure in Cartesian coordinates.

Our goal is to fuse the probability densities $f_{02}(\mathbf{x}_{02})$ and $f_{12}(\mathbf{x}_{12})$ in order to obtain the best estimate of the position of point 2 relative to point 0. The true position of this point is denoted as $\tilde{\mathbf{x}}_{02}$, whereas $\mathbf{x}_{02}$ can be any hypothetical value of it.

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Since

$$\mathbf{x}_{12} = \mathbf{x}_{02} - \mathbf{x}_{01}, \tag{1}$$

the fused density will be of the form

$$f_3(\mathbf{x}_{02}) \doteq \alpha \cdot f_1(\mathbf{x}_{02}) \cdot f_2(\mathbf{x}_{02}) \tag{2}$$

where for fixed $\mathbf{x}_{01} = \tilde{\mathbf{x}}_{01}$

$$f_1(\mathbf{x}_{02}) \doteq f_{02}(\mathbf{x}_{02}) \text{ and } f_2(\mathbf{x}_{02}) \doteq f_{12}(\mathbf{x}_{02} - \tilde{\mathbf{x}}_{01}) \tag{3}$$

and α is a scaling parameter that ensures that $f_3(\mathbf{x}_{02})$ is a probability density.

If the measurement device is accurate, we would expect either the mean or the mode (or both) of $f_{i2}(\mathbf{x}_{i2})$ to be at or near the true value $\tilde{\mathbf{x}}_{i2}$.

Eq. (2) can be viewed as a form of Bayesian fusion wherein one of the measurement probability densities is considered as the prior, and the other is a likelihood, and their product results in a posterior probability density. That is, in general Bayes' rule states that

$$f(\theta,x) = f(\theta\,|\,x)\,f(x) = f(x\,|\,\theta)\,f(\theta).$$

Therefore,

$$f(\theta\,|\,x) = \frac{f(x\,|\,\theta)\,f(\theta)}{f(x)}.$$

In our case $\theta = \mathbf{x}_{02}$, $f(\theta\,|\,x) = f_3(\mathbf{x}_{02})$, $f(\theta) = f_1(\mathbf{x}_{02})$, and $f(\theta\,|\,x)/f(x) = \alpha\, f_2(\mathbf{x}_{02})$.

Our problem has slightly more structure than the general Bayesian fusion scenario, since α can be computed explicitly as

$$\alpha^{-1} = (f_{02} * \bar{f}_{12})(\tilde{\mathbf{x}}_{01})$$

where $*$ denotes the convolution

$$(f_1 * f_2)(\mathbf{x}) \doteq \int_{\mathbb{R}^2} f_1(\mathbf{y})\, f_2(\mathbf{x}-\mathbf{y})\, d\mathbf{y}$$

and $\bar{f}(\mathbf{x}) \doteq f(-\mathbf{x})$.

For sensing modalities that collect position data in polar coordinates,

$$\mathbf{x} = [r\cos\phi, r\sin\phi]^T,$$

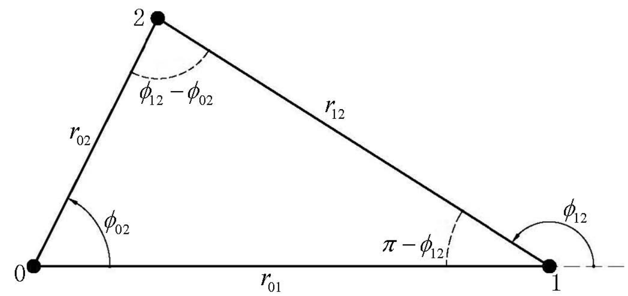

FIGURE 1. Description of Variables

a natural representation of measurement probabilities is the Fourier-Bessel description

$$f(\mathbf{x}) = f(r,\phi) = \sum_{n=-\infty}^{\infty} e^{in\phi} \int_0^{\infty} \hat{f}_n(p) J_n(pr)\, p\, dp. \tag{4}$$

This consists of a Fourier series in the ϕ direction and a Hankel transform in the r direction, and the functions $\{\hat{f}_n(p)\,|\,n \in \mathbb{Z}, p \in \mathbb{R}_{\geq 0}\}$ define $f(\mathbf{x})$, and can be recovered from $f(\mathbf{x})$ using the formula of measurement probabilities is the Fourier-Bessel description[1]

$$\hat{f}_{n''}(p'') = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} f(r,\phi) e^{-in''\phi} J_{n''}(p''r)\, r\, dr\, d\phi \tag{5}$$

In polar coordinates (2) and (3) give

$$f_3(r_{02},\phi_{02}) = \alpha \cdot f_{02}(r_{02},\phi_{02}) \cdot f_{12}(r_{12},\phi_{12}). \tag{6}$$

By manipulating expressions in Watson's classic work on Bessel functions [5], it can be shown that

$$e^{in'\phi_{12}} J_{n'}(p' r_{12}) =$$

$$e^{in'\phi_{02}} \cdot \sum_{m=-\infty}^{\infty} J_{n'+m}(p' r_{02}) J_m(p' r_{01}) e^{-im\phi_{02}} \tag{7}$$

where in the context of our problem, $r_{01} = \tilde{r}_{01}$ is the true fixed distance between the sensor positions, and $\phi_{01} = 0$ since both sensors lie on the x axis. Here $J_{n'}(p' r_{12})$ denotes Bessel functions of the first kind.

[1]The reason for using n'' and p'' here rather than n and p will become clear later.

Therefore, if we know the sensor measurements of point 2 taken from location 1, i.e., $f_{12}(r_{12},\phi_{12})$, then we can convert this to a representation at point 0 using (7) to fuse the two measurements.

To do this, we use (4) and

$$f_{02}(r_{02},\phi_{02}) = \sum_{n=-\infty}^{\infty} e^{in\phi_{02}} \int_0^{\infty} \hat{f}_n^{(02)}(p) J_n(p\,r_{02}) p\,dp$$

and

$$f_{12}(r_{12},\phi_{12}) = \sum_{n'=-\infty}^{\infty} e^{in'\phi_{12}} \int_0^{\infty} \hat{f}_{n'}^{(12)}(p') J_{n'}(p'\,r_{12}) p'\,dp'$$

Substituting (7) into the second of these and using (6) expresses $f_3(r_{02},\phi_{02})$ as

$$\begin{aligned} &f_3(r_{02},\phi_{02}) = \\ &\alpha \cdot \sum_{n=-\infty}^{\infty} e^{in\phi_{02}} \int_0^{\infty} \hat{f}_n^{(02)}(p) J_n(p\,r_{02}) p\,dp \sum_{n'=-\infty}^{\infty} e^{in'\phi_{02}} \\ &\cdot \int_0^{\infty} \hat{f}_{n'}^{(12)}(p') \sum_{m=-\infty}^{\infty} J_{n'+m}(p'\,r_{02}) J_m(p'\,r_{01}) e^{-im\phi_{02}} p'\,dp' \end{aligned} \tag{8}$$

We compute $\hat{f}_{n''}^{(3)}(p'')$ with (5), where $f(r,\phi) = f_3(r_{02},\phi_{02})$. The result is

$$\begin{aligned} &\hat{f}_{n''}^{(3)}(p'') = \\ &\frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} f_3(r_{02},\phi_{02}) e^{-in''\phi_{02}} J_{n''}(p'' r_{02}) r_{02}\,dr_{02}\,d\phi_{02} \\ &= \alpha \delta_{n+n',n''+m} \sum_{n=-\infty}^{\infty} \sum_{n'=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \int_0^{\infty} \int_0^{\infty} \hat{f}_n^{(02)}(p) \hat{f}_{n'}^{(12)}(p') \cdot \\ &J_m(p'\,r_{01}) \int_0^{\infty} J_n(p\,r_{02}) J_{n''}(p'' r_{02}) J_{n'+m}(p'\,r_{02}) r_{02}\,dr_{02}\,p p'\,dp\,dp' \\ &= \alpha \cdot \sum_{n=-\infty}^{\infty} \sum_{n'=-\infty}^{\infty} \int_0^{\infty} \int_0^{\infty} \hat{f}_n^{(02)}(p) \hat{f}_{n'}^{(12)}(p') J_{n+n'-n''}(p'\,r_{01}) \cdot \\ &\int_0^{\infty} J_n(p\,r_{02}) J_{n''}(p'' r_{02}) J_{n+2n'-n''}(p'\,r_{02}) r_{02}\,dr_{02}\,p p'\,dp\,dp' \end{aligned} \tag{9}$$

This can be simplified as

$$\begin{aligned} &\hat{f}_{n''}^{(3)}(p'') = \\ &\alpha \cdot \sum_{n=-\infty}^{\infty} \sum_{n'=-\infty}^{\infty} \int_0^{\infty} \int_0^{\infty} \hat{f}_n^{(02)}(p) \hat{f}_{n'}^{(12)}(p') J_{n+n'-n''}(p'\,r_{01}) \cdot \\ &\begin{pmatrix} n & n'' & n+2n'-n'' \\ p & p'' & p' \end{pmatrix} p p'\,dp\,dp' \end{aligned} \tag{10}$$

where we denote the three-term Bessel integral as

$$\begin{pmatrix} m & n & l \\ a & b & c \end{pmatrix} \doteq \int_0^{\infty} J_m(ar) J_n(br) J_l(cr)\, r\,dr. \tag{11}$$

Remarkably, such integrals appear not to have been studied in much detail in the classical literature, and only recently have been investigated [3]. In order to facilitate the computation of $\hat{f}_{n''}^{(3)}(p'')$, and hence the resulting posterior density $f_3(r_{02},\phi_{02})$, we investigate efficient ways to compute these integrals.

There are some obvious symmetries and redundancies in the integral in (11). For example, columns can be permutated simply because changing the order of the scalar multiplication of the Bessel functions in the integral will not change the resulting value. Moreover, since $a,b,c > 0$ we can change the variables of integration as follows. Let $r' = cr$, we can get $dr' = c\,dr$, $r = c^{-1}r'$ and $dr = c^{-1}dr'$. Substituting into (11) then gives

$$\begin{aligned} \begin{pmatrix} m & n & l \\ a & b & c \end{pmatrix} &= \frac{1}{c^2} \int_0^{\infty} J_m\left(\frac{a}{c}r'\right) J_n\left(\frac{b}{c}r'\right) J_l\left(r'\right) r'\,dr' \\ &= \frac{1}{c^2} \begin{pmatrix} m & n & l \\ \frac{a}{c} & \frac{b}{c} & 1 \end{pmatrix} \end{aligned} \tag{12}$$

when $c = 0$ and $m = n$

$$\begin{aligned} \begin{pmatrix} m & n & l \\ a & b & 0 \end{pmatrix} &= \int_0^{\infty} J_m(ar) J_n(br) J_l(0) r\,dr \\ &= \delta_{l,0} \frac{\delta(a-b)}{a}, \end{aligned} \tag{13}$$

where $\delta_{l,0}$ denotes Kronecker delta function and $\delta(a-b)$ is Dirac delta function.

In the following sections we will make use of the classical relations [5]

$$J'_m(x) = \frac{1}{2}[J_{m-1}(x) - J_{m+1}(x)] \tag{14}$$

$$J_m(x) = \frac{x}{2m}[J_{m+1}(x) + J_{m-1}(x)] \tag{15}$$

$$J_{-m}(x) = (-1)^m J_m(x) \tag{16}$$

$$(x^m J_m(x))' = x^m J_{m-1}(x) \tag{17}$$

and the related fact

$$\begin{aligned}
&\frac{d[(ax)^m J_m(ax)]}{da} \\
&= m(ax)^{m-1} x J_m(ax) + (ax)^m J'(ax)x \\
&= m(ax)^{m-1} x \frac{ax}{2m}[J_{m+1}(ax) + J_{m-1}(ax)] \\
&\quad + (ax)^m \frac{1}{2}[J_{m-1}(ax) - J_{m+1}(ax)]x \\
&= a^m x^{m+1} J_{m-1}(ax)\,.
\end{aligned}$$

These will allow us to develop recurrence relations for the three-Bessel-function integrals based on the classical recurrence relations for the Bessel functions themselves. To start the recurrence relations we seek closed-form expressions for some special cases, which is the subject of the following section.

STARTING VALUES FOR RECURSIONS

In special cases the three-Bessel-function integral in (11) have closed-form solutions which can be used to initiate recursions for computing the other values. In this section we focus on these special cases, and then in Section we develop recurrence relations.

We begin by observing the known closed-form integral [2]

$$\begin{aligned}
f(a,b,c,\alpha) &\doteq \int_0^\infty J_\alpha(ar)J_\alpha(br)J_\alpha(cr)r^{1-\alpha}dr \qquad (18)\\
&= \frac{[c^2-(a-b)^2]^{\alpha-\frac{1}{2}}[(a+b)^2-c^2]^{\alpha-\frac{1}{2}}}{2^{3\alpha-1}\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})(abc)^\alpha}.
\end{aligned}$$

If we set $\alpha = 0$ in (18) then

$$\begin{aligned}
f(a,b,c,0) &= \begin{pmatrix} 0 & 0 & 0 \\ a & b & c \end{pmatrix} \\
&= \int_0^\infty J_0(ar)J_0(r)J_0(cr)rdr \qquad (19)\\
&= \frac{[c^2-(a-b)^2]^{-\frac{1}{2}}[(a+b)^2-c^2]^{-\frac{1}{2}}}{2^{-1}\sqrt{\pi}\Gamma(\frac{1}{2})}\,,
\end{aligned}$$

which is a closed-form expression for a particular three-Bessel-function integral. Moreover, if we set $\alpha = 1$, then

$$\begin{aligned}
f(a,b,c,1) &\doteq \int_0^\infty J_1(ar)J_1(br)J_1(cr)dr \\
&= \int_0^\infty \frac{ar}{2}[J_2(ar)+J_0(ar)]J_1(br)J_1(cr)dr \qquad (20)\\
&= \frac{a}{2}\begin{pmatrix} 2 & 1 & 1 \\ a & b & c \end{pmatrix} + \frac{a}{2}\begin{pmatrix} 0 & 1 & 1 \\ a & b & c \end{pmatrix}.
\end{aligned}$$

If we take the partial derivative of this with respect to a, and use the properties of Bessel functions listed earlier, then

$$\begin{aligned}
\frac{\partial f(a,b,c,1)}{\partial a} &= \frac{1}{2}\int_0^\infty [J_0(ar)-J_2(ar)]J_1(br)J_1(cr)rdr \\
&= \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ a & b & c \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 2 & 1 & 1 \\ a & b & c \end{pmatrix}
\end{aligned} \qquad (21)$$

By combining the above equations as $(20)+(21)*a$, we get the closed-form expression

$$\begin{aligned}
\frac{f + a\frac{\partial f}{\partial a}}{a} &= \begin{pmatrix} 0 & 1 & 1 \\ a & b & c \end{pmatrix} \\
&= \frac{a^3+bc^2-ab^2}{2\sqrt{\pi}\Gamma(\frac{3}{2})(abc)\sqrt{a^4-2a^2b^2+4abc^2-c^4+b^4}}\,.
\end{aligned} \qquad (22)$$

And similarly, by combining the above equations as $(20)-(21)*a$, we get the closed-form expression

$$\begin{aligned}
\frac{f - a\frac{\partial f}{\partial a}}{a} &= \begin{pmatrix} 2 & 1 & 1 \\ a & b & c \end{pmatrix} \\
&= \frac{6abc^2-2c^4+2b^4-2a^2b^2}{2^2\sqrt{\pi}\Gamma(\frac{3}{2})(a^2bc)\sqrt{a^4-2a^2b^2+4abc^2-c^4+b^4}}\,.
\end{aligned} \qquad (23)$$

Now we set $\alpha = 2$, and evaluate

$$f(a,b,c,2) = \int_0^\infty J_2(ar)J_2(br)J_2(cr)r^{-1}dr\,, \qquad (24)$$

which is also a special case of the closed-form expression in (18).

Multiplied by a^2 on both sides of the equation (24) gives

$$a^2 f(a,b,c,2) = \int_0^\infty (ar)^2 J_2(ar)J_2(br)J_2(cr)r^{-3}dr\,. \qquad (25)$$

We then take the partial derivative with respect to a for both sides to give

$$2af + a^2 f' = \int_0^\infty a^2 r^3 J_1(ar)J_2(br)J_2(cr)r^{-3}dr \qquad (26)$$

where $'$ denotes $\partial/\partial a$. Then dividing by a on both sides of the equation (26) gives

$$2f + af' = \int_0^\infty arJ_1(ar)J_2(br)J_2(cr)r^{-1}dr\,. \qquad (27)$$

Then taking the partial derivation with respect to a on both sides of the Equation(27) again, we can get

$$2f' + f' + af'' = \int_0^\infty ar^2 J_0(ar)J_2(br)J_2(cr)r^{-1}dr \qquad (28)$$

$$\frac{3f' + af''}{a} = \begin{pmatrix} 0 & 2 & 2 \\ a & b & c \end{pmatrix}$$

More generally, if we let f be defined as $f(a,b,c,\alpha) \doteq \int_0^\infty J_\alpha(ar)J_\alpha(br)J_\alpha(cr)r^{1-\alpha}dr$, and we multiply by a^α then we get

$$a^\alpha f(a,b,c,\alpha) = \int_0^\infty (ar)^\alpha J_\alpha(ar)J_\alpha(br)J_\alpha(cr)r^{1-2\alpha}dr. \qquad (29)$$

Taking the partial of (29) with respect to a, and letting $A_0 = a^\alpha f(a,b,c,\alpha)$ the result is

$$\frac{\partial A_0}{\partial a} = \int_0^\infty a^\alpha r^{\alpha+1} J_{\alpha-1}(ar)J_\alpha(br)J_\alpha(cr)r^{1-2\alpha}dr. \qquad (30)$$

Dividing by a on both sides of (30) and letting $A_1 \doteq \frac{\partial A_0}{\partial a}a^{-1}$, we get

$$A_1 = \int_0^\infty (ar)^{\alpha-1} J_{\alpha-1}(ar)J_\alpha(br)J_\alpha(cr)r^{1-2\alpha+2}dr. \qquad (31)$$

Moreover, taking the partial derivative with respect to a and dividing (29) α times, we can get

$$\begin{pmatrix} 0 & \alpha & \alpha \\ a & b & c \end{pmatrix} = A_\alpha = \frac{\partial A_{\alpha-1}}{\partial a}a^{-1}. \qquad (32)$$

Doing the same calculations on both a and b in (24),

$$abf = \int_0^\infty (ar)J_2(ar)(br)J_2(br)J_2(cr)r^{-3}dr \qquad (33)$$

then taking the derivative of a for both sides of (33),

$$bf + ab\frac{\partial f}{\partial a} = \int_0^\infty ar^2 J_1(ar)(br)J_2(br)J_2(cr)r^{-3}dr. \qquad (34)$$

Dividing by a on both sides of the equation (34),

$$\frac{bf}{a} + b\frac{\partial f}{\partial a} = \int_0^\infty J_1(ar)(br)J_2(br)J_2(cr)r^{-1}dr. \qquad (35)$$

Then taking the derivative of b for both sides of the Equation(35),

$$\frac{f}{a} + \frac{b}{a}\frac{\partial f}{\partial b} + \frac{\partial f}{\partial a} + b\frac{\partial f}{\partial a \partial b} = \int_0^\infty J_1(ar)br^2 J_1(br)J_2(cr)r^{-1}dr. \qquad (36)$$

Dividing by b on both sides of the equation (36),

$$\frac{f}{ab} + \frac{1}{a}\frac{\partial f}{\partial b} + \frac{1}{b}\frac{\partial f}{\partial a} + \frac{\partial f}{\partial a \partial b} = \int_0^\infty J_1(ar)J_1(br)J_2(cr)rdr = \begin{pmatrix} 1 & 1 & 2 \\ a & b & c \end{pmatrix}. \qquad (37)$$

We want $\int_0^\infty J_\alpha(ar)J_\alpha(br)J_\alpha(cr)r^{1-\alpha}dr$ to change to $\int_0^\infty J_m(ar)J_n(br)J_l(cr)rdr$ by taking derivatives.

$$(abc)^\alpha f = \int_0^\infty (ar)^\alpha J_\alpha(ar)(br)^\alpha J_\alpha(br)(cr)^\alpha J_\alpha(cr)r^{1-4\alpha}dr \qquad (38)$$

let $B_0 = \frac{\partial[(abc)^\alpha f]}{\partial a}$, $m = \alpha - \alpha_1$, $n = \alpha - \alpha_2$, $l = \alpha - \alpha_3$

$$B_1 = \frac{\partial B_0}{\partial a}a^{-1}$$

$$\vdots$$

$$B_{\alpha_1} = \frac{\partial B_{\alpha_1 - 1}}{\partial a}a^{-1} = \int_0^\infty r^{\alpha+\alpha_1} J_m(ar)(br)^\alpha J_\alpha(br)(cr)^\alpha J_\alpha(cr)r^{1-4\alpha}dr$$

$$B_{\alpha_1+1} = \frac{\partial B_{\alpha_1}}{\partial b}b^{-1}$$

$$\vdots$$

$$B_{\alpha_1+\alpha_2} = \frac{\partial B_{\alpha_1+\alpha_2-1}}{\partial b}b^{-1} = \int_0^\infty r^{\alpha+\alpha_1} J_m(ar)r^{\alpha+\alpha 2} J_n(br)(cr)^\alpha J_\alpha(cr)r^{1-4\alpha}dr$$

$$B_{\alpha_1+\alpha_2+1} = \frac{\partial B_{\alpha_1+\alpha_2}}{\partial c}c^{-1}$$

$$\vdots$$

$$B_{\alpha_1+\alpha_2+\alpha_3} = \frac{\partial B_{\alpha_1+\alpha_2+\alpha_3-1}}{\partial c}c^{-1} = \int_0^\infty r^{\alpha+\alpha_1} J_m(ar)r^{\alpha+\alpha_2} J_n(br)r^{\alpha+\alpha_3} J_l(cr)r^{1-4\alpha}dr$$

if $\alpha + \alpha_1 + \alpha + \alpha_2 + \alpha + \alpha_3 = 4\alpha$, $m + n + l = 2\alpha$, we can get

$$B_{\alpha_1+\alpha_2+\alpha_3} = \begin{pmatrix} m & n & l \\ a & b & c \end{pmatrix} \qquad (39)$$

The closed-form expressions for special three-Bessel-function integrals computed in this section can be used as the initial values in the recursive computation of other integrals of this form. The recurrence relations used in these recursive computations are derived in the following section.

RECURRENCE RELATIONS

We define

$$g(a,b,c,m,n,l) \doteq \int_0^\infty J_m(ar)J_n(r)J_l(cr)dr. \tag{40}$$

Then

$$\begin{aligned} & g(a,b,c,m,n,l) \\ & = \int_0^\infty \frac{ar}{2m}[J_{m+1}(ar)+J_{m-1}(ar)]J_n(br)J_l(cr)dr \\ & = \frac{a}{2m}\left[\begin{pmatrix} m+1 & n & l \\ a & b & c\end{pmatrix} + \begin{pmatrix} m-1 & n & l \\ a & b & c\end{pmatrix}\right], \end{aligned} \tag{41}$$

and if we do the same with b instead of a, we can get

$$\begin{aligned} & g(a,b,c,m,n,l) \\ & = \int_0^\infty \frac{br}{2n}J_m(ar)[J_{n+1}(br)+J_{n-1}(br)]J_l(cr)dr \\ & = \frac{b}{2n}\left[\begin{pmatrix} m & n+1 & l \\ a & b & c\end{pmatrix} + \begin{pmatrix} m & n-1 & l \\ a & b & c\end{pmatrix}\right], \end{aligned} \tag{42}$$

and with c,

$$\begin{aligned} & g(a,b,c,m,n,l) \\ & = \int_0^\infty \frac{cr}{2l}J_m(ar)J_n(br)[J_{l+1}(cr)+J_{l-1}]dr \\ & = \frac{c}{2l}\left[\begin{pmatrix} m & n & l+1 \\ a & b & c\end{pmatrix} + \begin{pmatrix} m & n & l-1 \\ a & b & c\end{pmatrix}\right]. \end{aligned} \tag{43}$$

We use integration by parts to compute

$$\begin{aligned} & \int_0^\infty J_m(ar)J_n(br)J_l(cr)dr \\ & = J_m(ar)J_n(br)J_l(cr)r\Big|_0^\infty \\ & \quad - \int_0^\infty \frac{\partial}{\partial r}[J_m(ar)J_n(br)J_l(cr)]rdr \\ & = 0 - \int_0^\infty [aJ_m'(ar)J_n(br)J_l(cr) \\ & + bJ_m(ar)J_n'(br)J_l(cr) + cJ_m(ar)J_n(br)J_l'(cr)]rdr \\ & = -\frac{1}{2}\left[a\begin{pmatrix} m-1 & n & l \\ a & b & c\end{pmatrix} - a\begin{pmatrix} m+1 & n & l \\ a & b & c\end{pmatrix}\right. \\ & \quad + b\begin{pmatrix} m & n-1 & l \\ a & b & c\end{pmatrix} - b\begin{pmatrix} m & n+1 & l \\ a & b & c\end{pmatrix} \\ & \quad \left. + c\begin{pmatrix} m & n & l-1 \\ a & b & c\end{pmatrix} - c\begin{pmatrix} m & n & l+1 \\ a & b & c\end{pmatrix}\right] \end{aligned} \tag{44}$$

the right side of (41) equal the right side of (44), then

$$\begin{aligned} & \left(\frac{a}{2m}-\frac{a}{2}\right)\begin{pmatrix} m+1 & n & l \\ a & b & c\end{pmatrix} + \left(\frac{a}{2m}+\frac{a}{2}\right)\begin{pmatrix} m-1 & n & l \\ a & b & c\end{pmatrix} \\ & = \frac{b}{2}\begin{pmatrix} m & n+1 & l \\ a & b & c\end{pmatrix} - \frac{b}{2}\begin{pmatrix} m & n-1 & l \\ a & b & c\end{pmatrix} \\ & + \frac{c}{2}\begin{pmatrix} m & n & l+1 \\ a & b & c\end{pmatrix} - \frac{c}{2}\begin{pmatrix} m & n & l-1 \\ a & b & c\end{pmatrix} \end{aligned} \tag{45}$$

We can get the similar result by computing the b and c.

$$\begin{aligned} & \left(\frac{b}{2n}-\frac{b}{2}\right)\begin{pmatrix} m & n+1 & l \\ a & b & c\end{pmatrix} + \left(\frac{b}{2n}+\frac{b}{2}\right)\begin{pmatrix} m & n-1 & l \\ a & b & c\end{pmatrix} \\ & = \frac{a}{2}\begin{pmatrix} m+1 & n & l \\ a & b & c\end{pmatrix} - \frac{a}{2}\begin{pmatrix} m-1 & n & l \\ a & b & c\end{pmatrix} \\ & + \frac{c}{2}\begin{pmatrix} m & n & l+1 \\ a & b & c\end{pmatrix} - \frac{c}{2}\begin{pmatrix} m & n & l-1 \\ a & b & c\end{pmatrix} \end{aligned} \tag{46}$$

$$\begin{aligned} & \left(\frac{c}{2l}-\frac{c}{2}\right)\begin{pmatrix} m & n & l+1 \\ a & b & c\end{pmatrix} + \left(\frac{c}{2l}+\frac{c}{2}\right)\begin{pmatrix} m & n & l-1 \\ a & b & c\end{pmatrix} \\ & = \frac{a}{2}\begin{pmatrix} m+1 & n & l \\ a & b & c\end{pmatrix} - \frac{a}{2}\begin{pmatrix} m-1 & n & l \\ a & b & c\end{pmatrix} \\ & + \frac{b}{2}\begin{pmatrix} m & n+1 & l \\ a & b & c\end{pmatrix} - \frac{b}{2}\begin{pmatrix} m & n-1 & l \\ a & b & c\end{pmatrix} \end{aligned} \tag{47}$$

Rewriting (45), (46) and (47) together in matrix form.

$$
\begin{aligned}
&\begin{pmatrix} \frac{a}{2}-\frac{a}{2m} & \frac{b}{2} & \frac{c}{2} \\ \frac{a}{2} & \frac{b}{2}-\frac{b}{2n} & \frac{c}{2} \\ \frac{a}{2} & \frac{b}{2} & \frac{c}{2}-\frac{c}{2l} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} m+1 & n & l \\ a & b & c \end{pmatrix} \\ \begin{pmatrix} m & n+1 & l \\ a & b & c \end{pmatrix} \\ \begin{pmatrix} m & n & l+1 \\ a & b & c \end{pmatrix} \end{pmatrix} \\
&= \begin{pmatrix} \frac{a}{2m}+\frac{a}{2} & \frac{b}{2} & \frac{c}{2} \\ \frac{a}{2} & \frac{b}{2n}+\frac{b}{2} & \frac{c}{2} \\ \frac{a}{2} & \frac{b}{2} & \frac{c}{2l}+\frac{c}{2} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} m-1 & n & l \\ a & b & c \end{pmatrix} \\ \begin{pmatrix} m & n-1 & l \\ a & b & c \end{pmatrix} \\ \begin{pmatrix} m & n & l-1 \\ a & b & c \end{pmatrix} \end{pmatrix}
\end{aligned} \tag{48}
$$

we compute the determinant of matrix, let

$$
M = \begin{pmatrix} \frac{a}{2}-\frac{a}{2m} & \frac{b}{2} & \frac{c}{2} \\ \frac{a}{2} & \frac{b}{2}-\frac{b}{2n} & \frac{c}{2} \\ \frac{a}{2} & \frac{b}{2} & \frac{c}{2}-\frac{c}{2l} \end{pmatrix}
$$

$$
\det(M) = \frac{abcl+abcm+abcn-abc}{8lmn} \tag{49}
$$

we want $\det(M) \neq 0$. $abcl+abcm+abcn-abc \neq 0$ and $mln \neq 0$. From the functions we know, if $l+m+n \neq 1$ and $m \neq 0, n \neq 0, l \neq 0$,must has M^{-1} make $M^{-1}M = I$.

let $N = \begin{pmatrix} \frac{a}{2m}+\frac{a}{2} & \frac{b}{2} & \frac{c}{2} \\ \frac{a}{2} & \frac{b}{2n}+\frac{b}{2} & \frac{c}{2} \\ \frac{a}{2} & \frac{b}{2} & \frac{c}{2l}+\frac{c}{2} \end{pmatrix}$

Multiplying M^{-1} on both sides of (48),

$$
\begin{pmatrix} \begin{pmatrix} m+1 & n & l \\ a & b & c \end{pmatrix} \\ \begin{pmatrix} m & n+1 & l \\ a & b & c \end{pmatrix} \\ \begin{pmatrix} m & n & l+1 \\ a & b & c \end{pmatrix} \end{pmatrix} = M^{-1}N \begin{pmatrix} \begin{pmatrix} m-1 & n & l \\ a & b & c \end{pmatrix} \\ \begin{pmatrix} m & n-1 & l \\ a & b & c \end{pmatrix} \\ \begin{pmatrix} m & n & l-1 \\ a & b & c \end{pmatrix} \end{pmatrix}. \tag{50}
$$

Let $D = M^{-1}N$, then

$D_{11} = \frac{2m}{l+m+n-1} - 1 \quad D_{12} = \frac{2bm}{a(l+m+n-1)}$

$D_{13} = \frac{2cm}{a(l+m+n-1)} \quad D_{21} = \frac{2an}{b(l+m+n-1)}$

$D_{22} = \frac{2n}{l+m+n-1} - 1 \quad D_{23} = \frac{2cn}{b(l+m+n-1)}$

$D_{31} = \frac{2al}{c(l+m+n-1)} \quad D_{32} = \frac{2bl}{c(l+m+n-1)}$

$D_{33} = \frac{2l}{l+m+n-1} - 1$

MODELS FOR POLAR MEASUREMENT PROBABILITIES

For ideal sensors, the measurement probability density models in Cartesian coordinates would be

$$
f_{k2}(\mathbf{x}_{k2}) = \delta(\mathbf{x}_{k2} - \tilde{\mathbf{x}}_{k2})
$$

for $k = 0,1$ where $\delta(\cdot)$ is the Dirac delta function for $\mathbb{R}^2$. We can convert these to polar coordinates as

$$
f_{k2}(r_{k2}, \phi_{k2}) = \frac{1}{r_{k2}} \delta(r_{k2} - \tilde{r}_{k2}) \delta(\phi_{k2} - \tilde{\phi}_{k2}).
$$

This can be expressed in a Fourier-Bessel expansion by substituting into (5) to get

$$
\hat{f}^{(k2)}_{n''}(p'') = \frac{1}{2\pi} e^{-in''\tilde{\phi}_{k2}} J_{n''}(p'' \tilde{r}_{k2}). \tag{51}
$$

Of course, if the sensors made exact measurements, there would be no need to perform fusion. As an opposite extreme, consider the case where a range sensor has absolutely no bearing information. This would correspond to the Figure 2 where the location of the observed object could equally be located at the two intersection points. This sort of range-only information corresponds to the pdf in real space with Fourier-Bessel coefficients (51) with only $n'' = 0$ and all other terms absent. As a

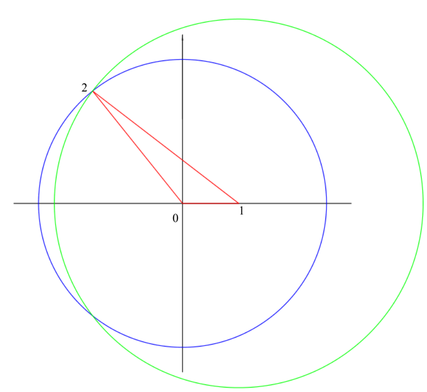

FIGURE 2. Fusion of Two Range-Only Measurements

more realistic model for the detection probability for the point $\tilde{\mathbf{x}}_{02}$ in the plane described in polar coordinates, (r_{02}, ϕ_{02}), we use a bandlimited approximation of the ideal Dirac delta. In other words, we set all of the coefficients in (51) to zero except for $\{\hat{f}_{n''}^{(k2)}(p'') \,|\, n'' = -1, 0, 1\}$. This has the effect of smearing out the delta function in the ϕ direction.

We set $\{\hat{f}_{n''}^{(k2)}(p'') \,|\, n'' = -1, 0, 1\}$ for the sensor information. The probability distribution function (PDF) of information when sensor at point 0 is shown in Figure 3, here the monitoring position is at point $(\tilde{r}, \tilde{\phi}) = (2, 0)$. The PDF of sensor information at point 1 is shown in Figure 4, where the distance between point 0 and point 1 $r_{01} = 3$. we can see PDF of direct multiplication of $f_{02}(\mathbf{x})$ and $f_{12}(\mathbf{x})$ in Figure 5. The PDF is computed by our information fusion algorithm that is shown in Figure 6. Comparing with the results of information fusion in Figure 5 and 6, the location of monitoring point is more accurate and clear in Figure 6. It proofs the useful and effective of Our information fusion algorithm.

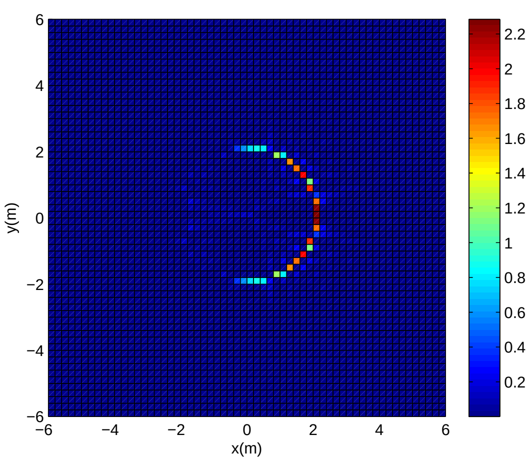

FIGURE 3. PDF of $f_{02}(\mathbf{x})$ when the sensor at point 0, the model for detection probability for the point $(\tilde{r}, \tilde{\phi}) = (2, 0)$.

CONCLUSIONS AND FUTURE WORK

We present a method to fuse probabilities corresponding to noisy measurement models in polar coordinates. At the core of this approach is the observation that integrals of products of three Bessel functions arise. We derive recurrence relations for these integrals in order to efficiently evaluate the fused distributions.

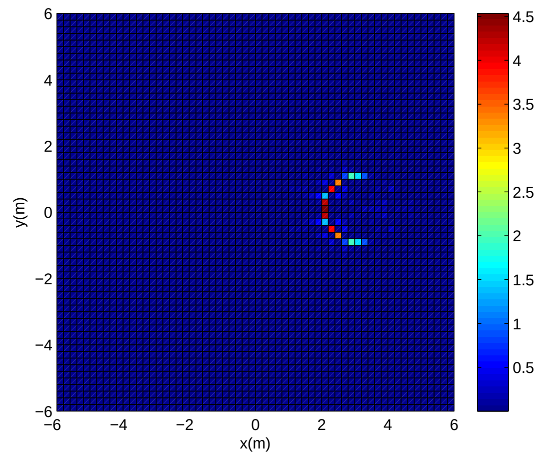

FIGURE 4. PDF of $f_{12}(\mathbf{x})$ when the sensor at point 1, the model for detection probability for the point $(\tilde{r}, \tilde{\phi}) = (2, 0)$, the distance between point 0 and point 1 $r_{01} = 3$.

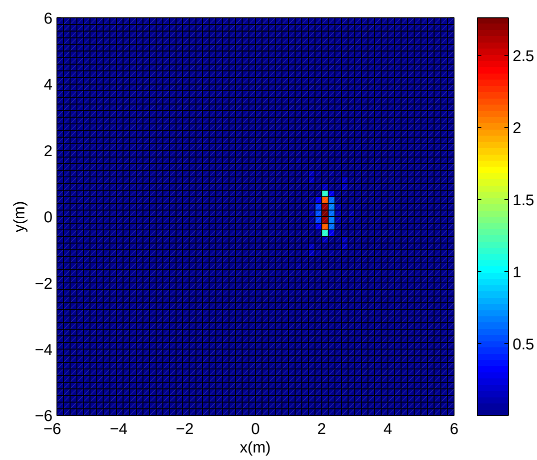

FIGURE 5. Fused PDF by direct multiplication of $f_{02}(\mathbf{x})$ and $f_{12}(\mathbf{x})$.

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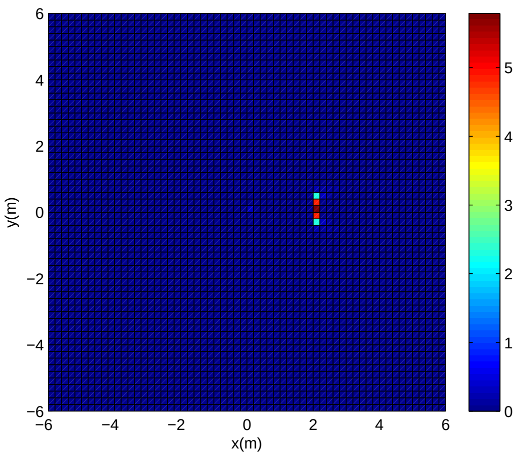

FIGURE 6. Resulting Fused PDF when $f_{12}(\mathbf{x})$ is shifted in the e_1 direction by 0.2. (In this case the two circles kiss at a point).

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