General Kinematic Synthesis Method for a Discretely Actuated Robotic Manipulator (D-ARM)

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***Abstract*— A "Discretely Actuated Robotic Manipulator", or "D-ARM", is any member of a class of robotic manipulators powered by actuators that have only discrete positional stable states such as solenoids. One of the most significant kinematic phenomena of D-ARMs is the discreteness of both input range and end-effector frames. The main characteristics of D-ARMs are: stability at each state without feedback loop; high task repeatability; mechanism simplicity; minimal supporting devices; low cost. These are strong advantages for manufacturing automation; mobile robot; space structure; micro/nano mechanism.**

The proposing design method is based on an incremental kinematic synthesis with a numerically obtained Jacobian matrix of a base-line manipulator and its generalized inverse matrix. The significance of this method is that it deals with a set of inverse kinematic problems on the Special Euclidean group in three space, $SE(3)$, instead of one on the Euclidean space, $\mathbb{R}^3$.

The conducted simulations demonstrate the feasibility of the synthesis method.

I. Introduction

A. Definition of D-ARM with Actuator Categorization

For robotic manipulation, actuators are key components. Actuators can be recognized as belonging to one of the following two kinematic categories: the first one is continuously position controllable and the other has only a finite number of discrete stable positions. The former actuator accepts a continuous range of input command values, but not the latter. Table I summarizes this categorization.

TABLE I
Actuator and Manipulator Categorization

Actuator	Example	Stable state = Input command	Manipulator
Continuous	Servomotor	Continuous range	C-ARM
Discrete	Solenoid, Pneumatic cylinder	Discrete range	D-ARM

We denote the former actuator as a "Continuous-Range-of-Motion Actuator" [1], or "Continuous Actuator", and the latter one as a "Discrete-Range-of-Motion Actuator", or "Discrete Actuator." We define a class of manipulators called the "Discretely Actuated Robotic Manipulator (D-ARM)" or "Discrete Arm" which is powered by discrete actuators. In particular, a "Binary Actuated Robotic Manipulator (B-ARM)" or "Binary Arm" is one with actuators that have only binary stable states. The categorization of D-ARM should be recognized as a generalization of the binary arm concept presented by Chirikjian [2]. Further, in contrast to D-ARM, let us call a manipulator a "Continuously Actuated Robotic Manipulator (C-ARM)", or "Continuous Arm", if the manipulator uses continuous actuators, as most standard robotic systems do.

B. Examples of D-ARM

Several examples of D-ARM are shown in Fig. **??**. In these examples, each actuator of a D-ARM is activated according to each digit of the assigned number which is an input command to the controller of the manipulator.

One of the most fundamental examples is shown in Fig. 1. The 2D (two dimensional) B-ARM that has three bi-stable actuators is activated according to a three-digit binary number. The left (right) actuator is associated with the most (least) significant bit of the binary number, and the center actuator corresponds to the middle one. The binary bit "1" ("0") means full extension (contraction) of the actuator. By changing the binary number given to the controller, one of eight ($= 2^3$) possible configurations of the 3-bit B-ARM can be selected, and this means that one of eight possible frames (elements of $SE(3)$: pairs of position and orientation) of the end-effector can be reached with a certain binary number by the B-ARM.

In the s-state (multi-state) actuator case, the end-effector frame assignment is done by a discrete number, which is an element of a general type of a discrete number system with the base of s. In the case of three-state actuators, the base of this discrete number system is three, e.g., "0", "1", or "2" will be used for each digit, instead of a binary digit (0 or 1) in the B-ARM case. One of the principle examples of the three-state D-ARM is shown in Fig. 2. This D-ARM can reach nine ($= 3^2$) frames by its end-effector. The discussion above is also applicable for the 3D case in a similar manner. In Fig. 3, 68.7 billion ($= 64^6$) end-frames are "reachable" by the 3D B-ARM example that consists of six stewart-type binary platforms.

One of the most significant kinematic phenomena of D-ARMs is the discreteness of both input range and end-effector frames as shown above.

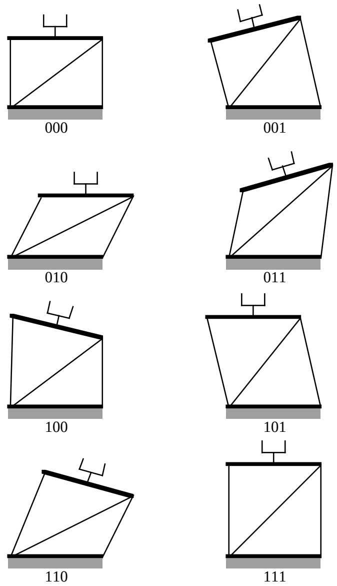


Fig. 1. An Example D-ARM (3-bit 2D Binary Parallel Platform)

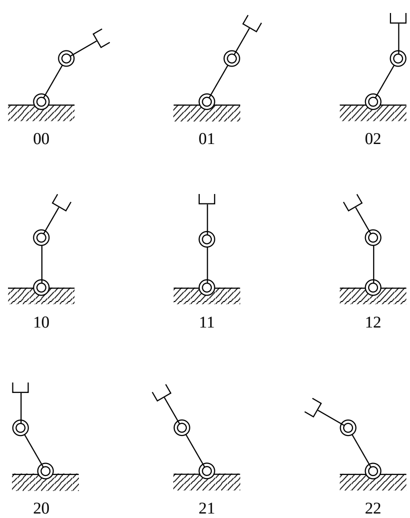


Fig. 2. An Example D-ARM (2-bit 2D Serial Arm with Three-state Actuators)

C. Advantages and Applications of D-ARM

While discrete actuators are the key components of D-ARMs, they are also widely used as stand-alone motion sources in various applications. The variety of those applications originates from the following significant characteristics of discrete actuators:

(1) Stability at each state without feedback loop
(2) High task repeatability

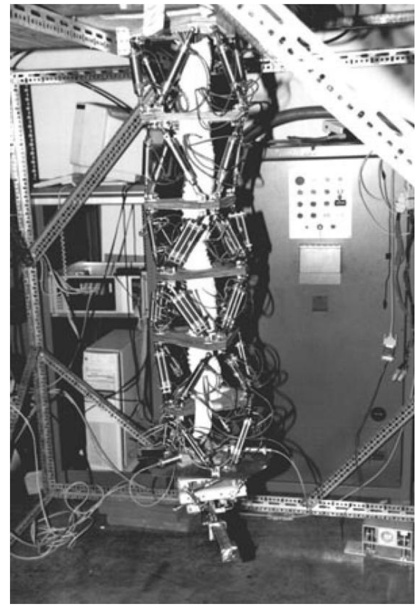


Fig. 3. An Example D-ARM (6 × 6-bit 3D Binary Parallel/Serial Hybrid Arm)

(3) Mechanism simplicity, including kinematic parameter adjustment (e.g., stroke stopper for a pneumatic cylinder)
(4) Minimal supporting devices, especially feedback systems
(5) Low cost and small volume due to (3) and (4)

Each state of a discrete actuator is mechanically stable because of simple internal mechanical constraints of parts, and the range of motion of a discrete actuator is also determined mechanically. The constraint is stable without an extensive, or even any, feedback loop (4). This results in the fundamental characteristics of discrete actuators: high stability (1) and high repeatability (2) which are mainly governed by the dimensional accuracy of the actuators parts. All of these characteristics of discrete actuators directly result in those of a D-ARM which is composed of discrete actuators.

To control robot motion continuously, such as a trajectory constrained task, continuous actuators are essential devices for the manipulator system. It should be noticed, however, that continuous actuation is not so essential for the teaching-playback work especially if the task is an ordinary pick-and-place one, since continuity of input range is not necessary for the playback phase. The key issue is the fact that, after each teaching process, the robot's actuators only go back and forth between several memorized "discrete" positions. Namely, continuous actuators are used as the robot's hardware in order to adapt a task by software programming for such a teaching phase.

Reconsidering continuous actuators from the point of view of cost efficiency, they seem to be "overkill" [3] when a task defines start/goal end-effector frames but the trajectory is less important as long as it is bounded. Conversely, discrete actuators could be a sufficient and cost effective solution for such a playback task if they have the ability to easily change kinematic parameters, such as stroke length for a cylinder.

The studies on "Sensor-less Manipulation" [4]show us that we do not have to use complicated systems whenever we encounter a technical problem. The approach in D-ARM research has commonality with the studies: we can design a robot without expensive continuous actuators if the desired task is not continuous motion control.

D. Scope of this paper

One of the most fundamental synthesis issues for manipulator design is to determine its kinematic parameters in order for reaching all the given desired frames.

Considering the kinematic synthesis of mechanical system with discrete actuation, it seems possible to design "intuitively" a system that can reach the desired end-frames if the number of the frames is small. In contrast, such a heuristic designing approach might not be feasible for the mechanism that has higher degrees of freedom (DOF) as "Hyper-redundant" manipulator [5].

In order to solve the latter design problem, several studies has been performed for B-ARM [2], [3], [6]. Such solutions, however, are intended to handle a positional kinematics only. The proposing kinematic synthesis process in this paper deals with the desired end-effector orientations as well as the desired positions, i.e., this paper is to propose the solution for the important synthesis problem of a D-ARM as follows:

> Given a D-ARM (base-line design) and finite sets of desired positions and orientations of the D-ARM's end-effector (desired frames: $\mathbf{H}_{ee} \in SE(N)$, where $N = 2(3)$ for 2D(3D) case), determine kinematic parameters (the vector $\mathbf{a}$) of the manipulator.

The design parameter $\mathbf{a}$ could be the link length for prismatic joints, or rotational angle for revolute joints of the manipulator. Note that this set of inverse kinematics problems becomes now on the Special Euclidean group, $SE(N)$, and that it is an extension of the previous B-ARM researches that discuss one on the Euclidean space, $\mathbb{R}^N$.

E. Related Works

The concept to use discrete actuators for robotic manipulators was, to the best of the author's knowledge, conceived in the 1960's [7], [8]

In the 1990's, the concrete concept of a binary manipulator as a new paradigm in robotics based on binary actuation was presented by Chirikjian [2]. Positional kinematic synthesis method for a binary manipulator in the 2D case has been presented [3]. The concept of workspace density in the context of efficient workspace generation was introduced [6], and the concept was applied to obtain a near optimal solution for the (positional) inverse kinematics problem [9].

A "Variable Geometry Truss", or VGT [10] is another example of a D-ARM if it employs discrete actuation concept. Particularly in the astronautical sciences [11], the VGT has been examined as a mechanism for manipulators and physical structures.

Some researchers in the Micro Electro Mechanical Systems, or MEMS, community have studied binary mechanisms [12], [13]. Applications differ from researcher to researcher in MEMS field; however, we see that those researchers take advantage of feasibility, reliability, and stability with binary actuation's simple mechanisms.

One of the most recent attempts to utilize the advantages of the discrete actuation can be found on the researches for robotic planetary explorers [14]. The mobile robot employs binary actuators to achieve a light weight and durable system with a simple and thus reliable controller.

II. FORMULATION

The flowchart of the proposing design process is shown in Fig. 4. The conceptual framework of the proposing synthesis method to solve the kinematic synthesis problem is based on an iterative computation with a numerically obtained Jacobian matrix of the given D-ARM and its weighted generalized inverse matrix.

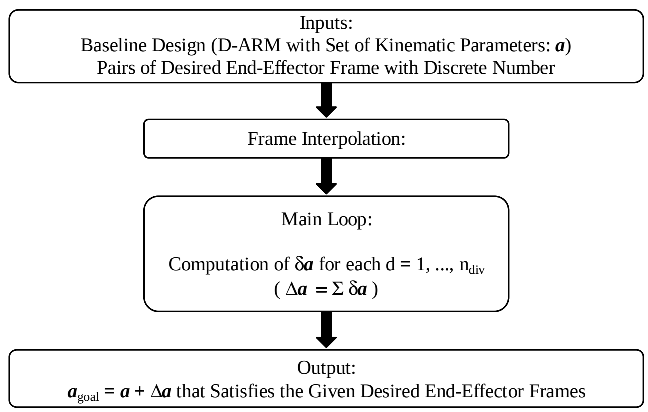


Fig. 4. Flowchart of the Proposing Kinematic Synthesis Method

In the iterative computation, the end-effector frame of a D-ARM is gradually shifted from its initial value to the given desired one, and at each step the kinematic parameters of the arm are updated in a computation that depends on the manipulator Jacobian matrix.

By using the forward kinematics of the manipulator, we can obtain a current end-effector frame of this manipulator with respect to any given joint vector at each iterative step. This frame can be denoted $\mathbf{H}_{ee\_k} \in SE(3)$ for $k = 1, \ldots, n_{frm}$, where n_{frm} denotes both the number of initial end frames and that of the desired frames. There is a one-to-one mapping between the frames through the assigned discrete number. n_{frm} also determines the kinematic conditions: sufficient, insufficient, or redundant. The computation for the sufficient number of the frames can be derived with the concept of the Configuration Tree "CT" [1]. In short, the sufficient number of frames is linear in the DOF of a D-ARM, despite the fact that the number of reachable frames is exponential in the DOF.

A sequence of frames, $\mathbf{H}_{k(d)} \in SE(3)$ for $d = 1, \ldots, n_{div}+1$ between the initial end frame, $\mathbf{H}_{k(1)}$, and the desired frame, $\mathbf{H}_{k(n_{div}+1)}$, can be obtained with the Rodrigues formula [15], where n_{div} is the number of the fixed-steps in the interpolation. The uniform distribution of the interpolated frames is guaranteed with a metric concept on rigid-body displacement [16]. The difference between any adjacent two frames, $\mathbf{H}_{k(d)}$ and $\mathbf{H}_{k(d+1)}$, can be small enough by choosing appropriate n_{div} for the frame sequences, i.e., the following condition holds:

$$\mathbf{H}_{k(d)}^{-1}\,\mathbf{H}_{k(d+1)} \approx \mathbf{I}\,, \tag{1}$$

for $d = 1, \ldots, n_{div}$, where $\mathbf{I}$ is the identity matrix.

The vector $\mathbf{a}$ represents a set of initial kinematic parameters. The vector $\mathbf{a}$ has the following structure for a general D-ARM with n_{st}-state actuators:

$$\mathbf{a} = \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_{n_{st}} \end{pmatrix}, \tag{2}$$

where $\mathbf{a}_i \in \mathbb{R}^{n_{act}}$ is the set of kinematic parameters of the i^{th} state of the discrete actuators, and n_{act} denotes the number of actuators. For instance, $\mathbf{a}$ has the following structure for a B-ARM:

$$\mathbf{a} = \begin{pmatrix} \mathbf{a}_{max} \\ \mathbf{a}_{min} \end{pmatrix}, \tag{3}$$

where $\mathbf{a}_{max}$ and $\mathbf{a}_{min}$ are the set of kinematic parameters with maximum and minimum values of the binary actuators, respectively.

In the main loop, focusing on a certain pair of initial end frame and desired frame with an assigned discrete number, we calculate $\delta\mathbf{a}_{(d)}$ such that the end-effector frame changes from $\mathbf{H}_{k(d)}$ to $\mathbf{H}_{k(d+1)}$ for a certain d. In order for the renewal of $\mathbf{a}_{(d+1)}$, a method to compute $\delta\mathbf{a}_{(d)}$ from $\mathbf{H}_{k(d)}$, $\mathbf{H}_{k(d+1)}$, and $\mathbf{a}_{(d)}$ is required as following:

$$\mathbf{a}_{(d+1)} \leftarrow \mathbf{a}_{(d)} + \delta\mathbf{a}_{(d)}\,. \tag{4}$$

The summation of $\delta\mathbf{a}_{(d)}$ composes $\Delta\mathbf{a}$ in the flowchart. This iterative computation can be done associated with the forward kinematic function of the manipulator as shown below.

The forward kinematics for a D-ARM can be written in the form:

$$\mathbf{H}_{k(d)} = \mathbf{F}_k(\mathbf{a}_{(d)})\,. \tag{5}$$

For simplicity, the subscripts of $\mathbf{F}_k$ and $\mathbf{a}_{(d)}$ will be omitted in the sequel. Letting $n_{parm} = n_{st} \cdot n_{act}$, the small change from $\mathbf{H}_{k(d)}$ to $\mathbf{H}_{k(d+1)}$ is described as following:

$$\mathbf{H}_{k(d)}^{-1}\,\mathbf{H}_{k(d+1)} = \mathbf{I} + \sum_{i=1}^{n_{parm}} \mathbf{F}(\mathbf{a})^{-1}\frac{\partial\mathbf{F}}{\partial a_i}\delta a_i\,. \tag{6}$$

Letting:

$$\mathbf{H} = \begin{pmatrix} \mathbf{Q} & \mathbf{b} \\ \mathbf{0}^T & 1 \end{pmatrix}, \tag{7}$$

where $\mathbf{Q} \in SO(3)$ and $\mathbf{b} \in \mathbb{R}^3$, we have in general:

$$\mathbf{H}^{-1}\frac{\partial\mathbf{H}}{\partial a_i} = \begin{pmatrix} \mathbf{Q}^T\dfrac{\partial\mathbf{Q}}{\partial a_i} & \mathbf{Q}^T\dfrac{\partial\mathbf{b}}{\partial a_i} \\ \mathbf{0}^T & 0 \end{pmatrix}. \tag{8}$$

Applying $(\cdot)^\vee$ operation on the screw matrix, $\mathbf{H}^{-1}\dfrac{\partial\mathbf{H}}{\partial a_i} \in se(3)$, and the skew-symmetric matrix, $\mathbf{Q}^T\dfrac{\partial\mathbf{Q}}{\partial a_i} \in so(3)$, we obtain:

$$\begin{aligned} \left(\mathbf{H}^{-1}\frac{\partial\mathbf{H}}{\partial a_i}\right)^\vee &= \begin{pmatrix} \mathbf{Q}^T\dfrac{\partial\mathbf{b}}{\partial a_i} \\ \left(\mathbf{Q}^T\dfrac{\partial\mathbf{Q}}{\partial a_i}\right)^\vee \end{pmatrix} \\ &\triangleq \begin{pmatrix} \mathbf{v}_i \\ \boldsymbol{\omega}_i \end{pmatrix}, \end{aligned} \tag{9}$$

Using this, we obtain the following for the latter term of Equation (6):

$$\left(\sum_{i=1}^{n_{parm}} \mathbf{F}(\mathbf{a})^{-1}\frac{\partial\mathbf{F}}{\partial a_i}\delta a_i\right)^\vee = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_{n_{parm}} \\ \boldsymbol{\omega}_1 & \boldsymbol{\omega}_2 & \ldots & \boldsymbol{\omega}_{n_{parm}} \end{pmatrix}\delta\mathbf{a}\,. \tag{10}$$

Letting $\delta\mathbf{x}_k$ be the small change through the sequence of interpolated frames for the k^{th} desired frame, together with Equation (6), we have:

$$\begin{aligned} \delta\mathbf{x}_k &= \left(\mathbf{H}_{k(d)}^{-1}\,\mathbf{H}_{k(d+1)} - \mathbf{I}\right)^\vee \\ &\triangleq \mathbf{G}_k\delta\mathbf{a}\,, \end{aligned} \tag{11}$$

where $\mathbf{G}_k$ is the Jacobian matrix for the k^{th} desired frame. Note that inside of the first parentheses is a screw matrix again.

The set of the obtained Equation (11) for every k at a certain d, can be combined into one big equation as following:

$$\begin{aligned} \begin{pmatrix} \delta\mathbf{x}_1 \\ \delta\mathbf{x}_2 \\ \vdots \\ \delta\mathbf{x}_{n_{frm}} \end{pmatrix} &= \begin{pmatrix} \mathbf{G}_1 \\ \mathbf{G}_2 \\ \vdots \\ \mathbf{G}_{n_{frm}} \end{pmatrix}\delta\mathbf{a} \\ &\triangleq \mathbf{G}\,\delta\mathbf{a}\,, \end{aligned} \tag{12}$$

where $\mathbf{G}$ is the "concatenated" Jacobian matrix of the given D-ARM at a certain d in the main loop. The rank of the system of equations in (12) can be changed by the kinematic case of the original manipulator, i.e., sufficient, insufficient, or redundant.

The final result of the main loop in the flowchart in Fig. 4 is the following equation from Equation (12):

$$\delta\mathbf{a} = \mathbf{G}^\dagger\,\delta\mathbf{x}\,, \tag{13}$$

where † denotes the weighted generalized inverse matrix. According to the case of manipulator kinematics, the equation either has a unique, no, or an infinite number of exact solutions in general. No matter which kinematic case the system belongs to, we can solve the system of equations by utilizing the idea of a weighted generalized inverse matrix.

The further expansion of this concept to the general multi-state D-ARM can be done in a same manner. Applying all the discussions above, similar but different size of the system equation would be obtained.

III. SIMULATION RESULTS

Several simulations are performed to demonstrate the feasibility of the algorithm discussed in the previous section. Through these simulations, it is found that the proposed synthesis algorithm is applicable to a wide variety of applications, and that the numerical errors were small enough and the computational time is short enough to design a real D-ARM even with a personal computer.

A typical example of the 2D sufficient case is shown in Fig. 5, and Fig. 6. Each link length of the manipulator is set as $1[m]$.

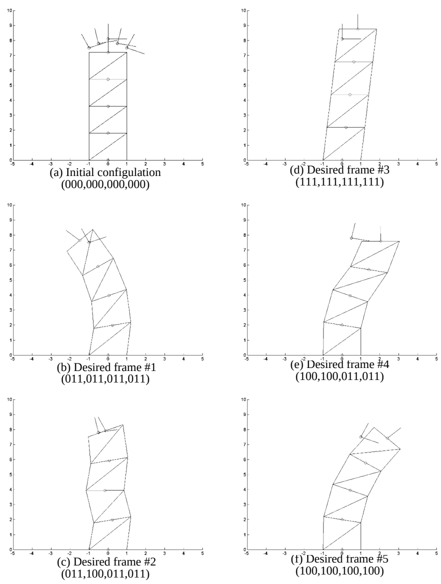


Fig. 5. Baseline Design Example (4-module 2D B-ARM, Sufficient case)

Fig. 5 shows the baseline design prior to synthesize. In particular, Sub-figure (a) shows the configuration with the binary number="000,000,000,000" as a reference configuration of the baseline design. Other sub-figures, (b)-(f), show both the given desired frames and arm configurations which corresponds to the assigned binary numbers.

Fig. 6 shows the final results of the kinematic synthesis process. The order of the position error was $10^{-7}[m]$ and that of the orientation error was $10^{-9}[deg]$ with 10^3 steps in the frame interpolation. The whole process was executed in the order of $10^2[sec]$ for this 2D case.

In the 3D sufficient case (no figures), the order of the position and the orientation error was less than $10^{-5}[m]$ and $10^{-5}[deg]$, respectively with 10^5 steps in the interpolation. The order of the computation time was $10^5[sec]$.

According to the complexity analysis, the order of n_{div} dominates the computation time for ordinary D-ARM system.

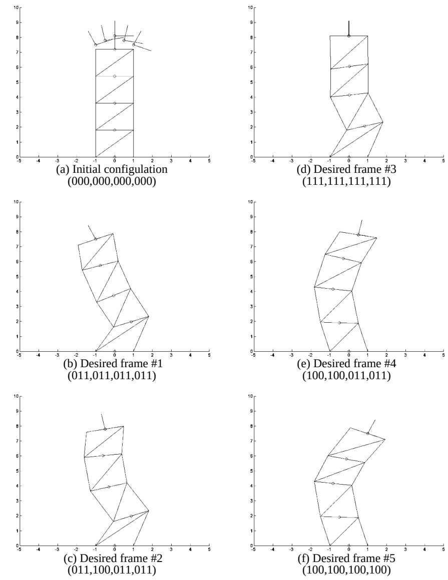
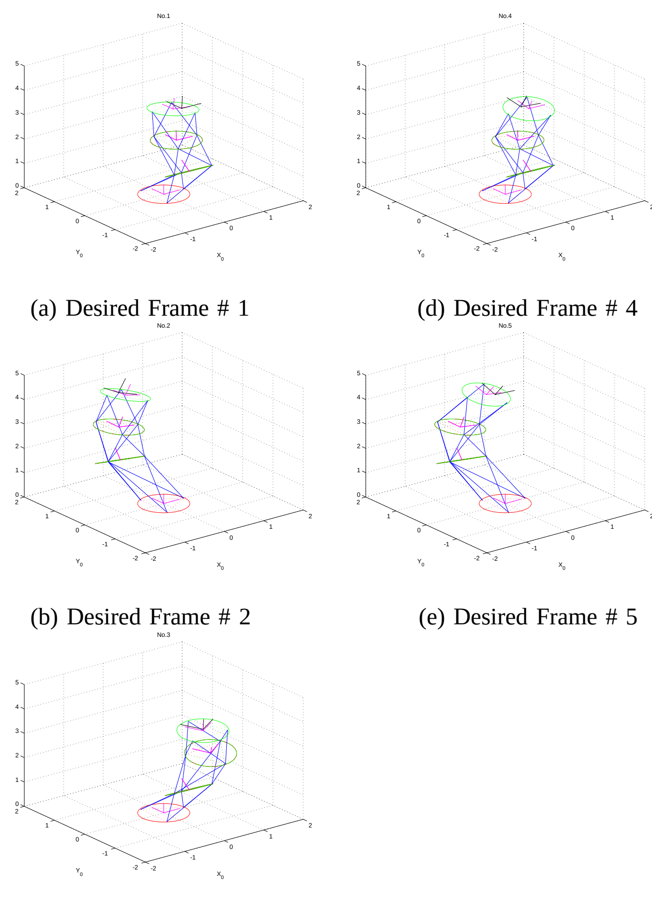


Fig. 6. Synthesis Result Example (4-module 2D B-ARM, Sufficient case)



(a) Desired Frame # 1

(d) Desired Frame # 4

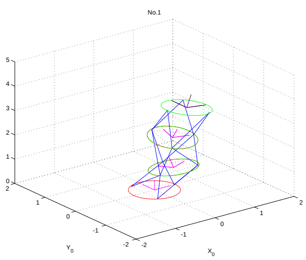
(b) Desired Frame # 2

(e) Desired Frame # 5

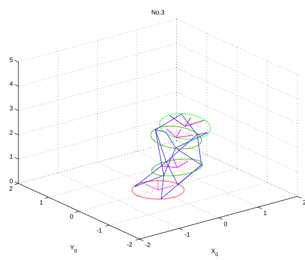
(c) Desired Frame # 3

Fig. 7. Synthesis Result Example (3-module 3D B-ARM, Insufficient case)

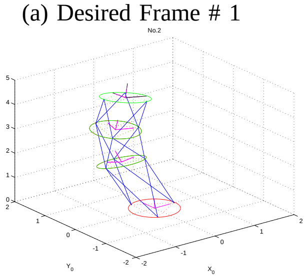
Fig. 7 shows the synthesis procedure result for an example case of the 3D insufficient kinematic condition. The regularization method [17] was applied for this example in order to cope with the instability of the numerical computation that is unavoidable due to machine ϵ. In this case, the "least squares solution" that minimizes the error vector can be obtained. The relationship between the stability and exactness of the solution is a trade-off, and finding an appropriate regularization parameter is required to design a well-balanced manipulator.



(a) Desired Frame # 1



(c) Desired Frame # 3

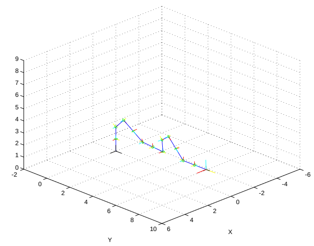


(b) Desired Frame # 2

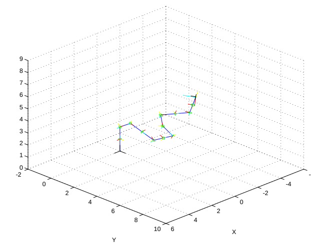
Fig. 8. Synthesis Result Example (3-module 3D B-ARM, Redundant case)

Fig. 8 shows the synthesis procedure result for 3D examples of redundant kinematic condition. In this kinematic condition, the evaluation function, $\left\|\delta\mathbf{a}\right\|$, which is the norm of kinematic parameter alteration, is minimized.The physical meaning of this is that the kinematic parameters of the designed D-ARM has a minimum alteration from the baseline design.

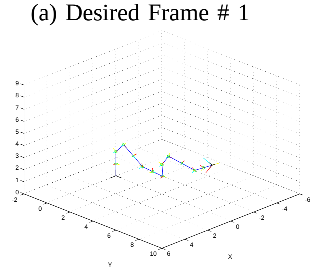
Another example application for a serial B-ARM with sufficient kinematic condition in the 3D case is shown in Fig. 9. The figure shows that the synthesis procedure has successfully designed this snake-like manipulator to reach the desired frames, as well.



(a) Desired Frame # 1



(c) Desired Frame # 3



(b) Desired Frame # 2

Fig. 9. Synthesis Result Example (12-DOF Serial 3D B-ARM, Sufficient case)

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